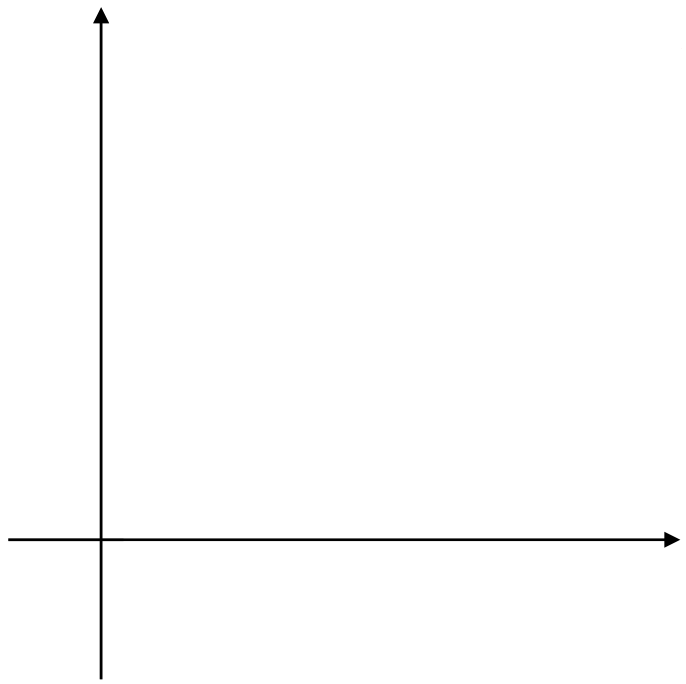
Functions

Variations of a function

Variations(prerequisites)

In grade 10 class, you’ve learned how to study the variations of a given function.



The graph of a function rises from left to right over an interval I.

Variations(prerequisites)

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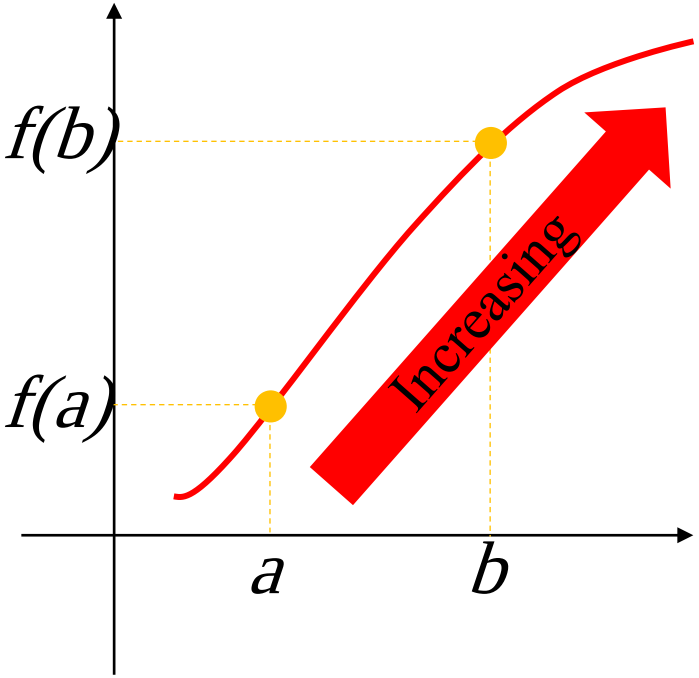

f is increasing function over an interval I:
For all a, b in I, if $a \leq b$ then $f(a) \leq f(b)$

The graph of a function rises from left to right over an interval I.

Variations(prerequisites)

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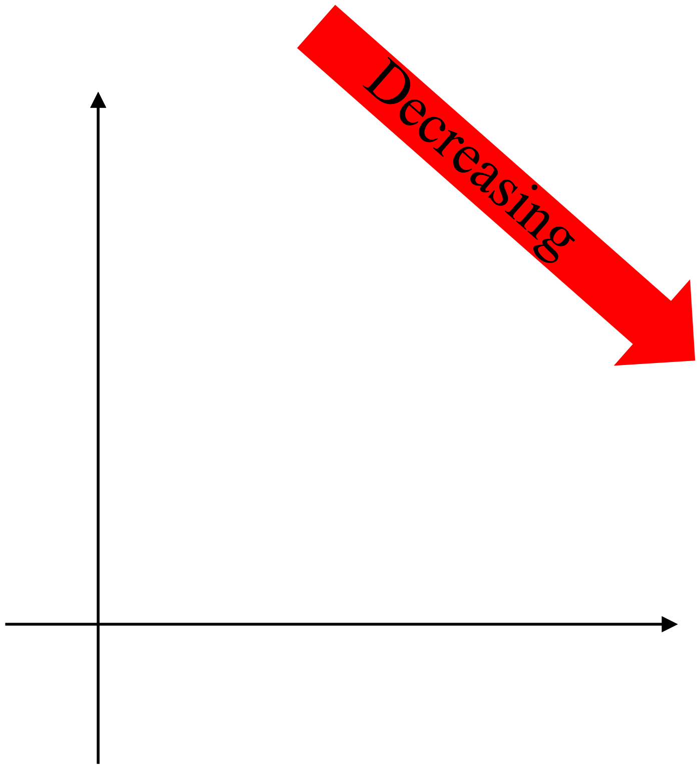

The graph of a function falls from left to right over an interval I.

Variations(prerequisites)

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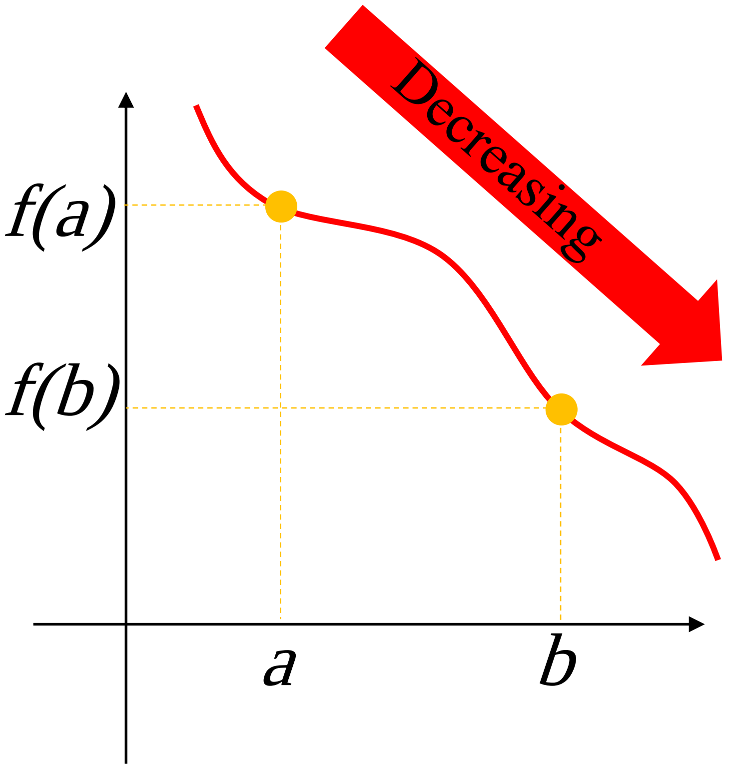
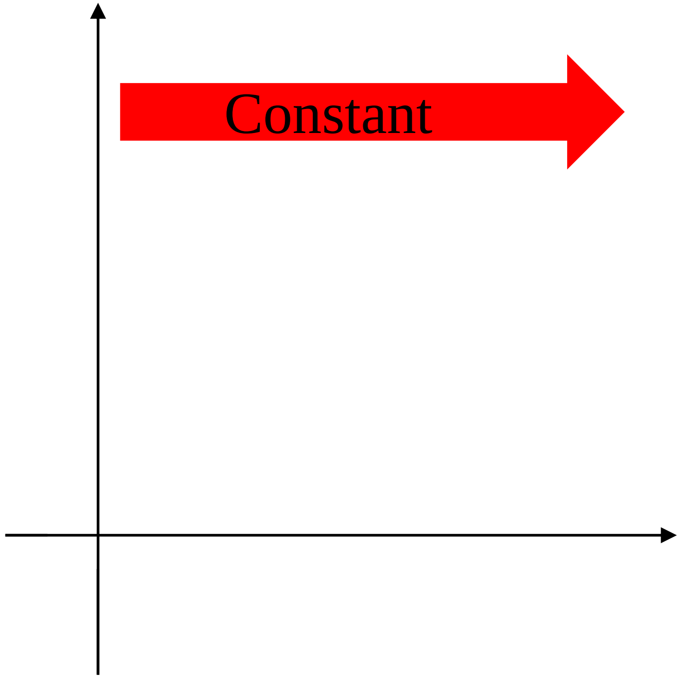


f is decreasing function over an interval I:
For all a, b in I, if $a \leq b$ then $f(a) \geq f(b)$

The graph of a function falls from left to right over an interval I.

Variations(prerequisites)

In grade 10 class, you've learned how to study the variations of a given function.



The graph of a function is a horizontal line over an interval I.

Variations(prerequisites)

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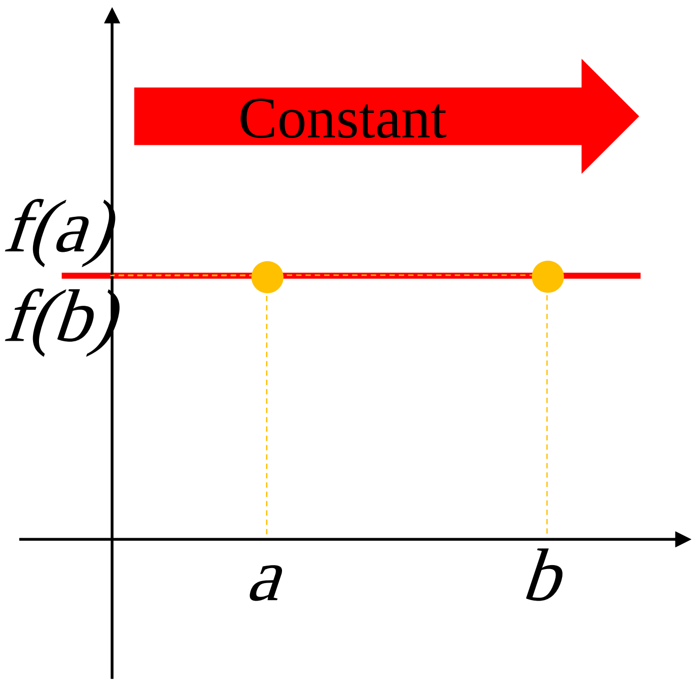


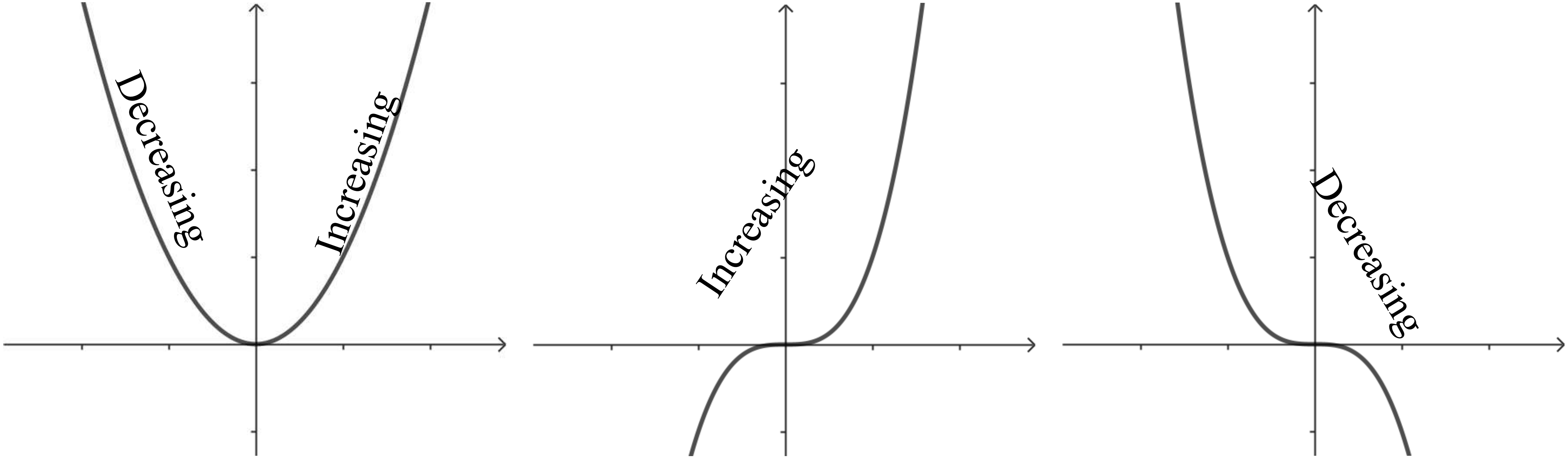
f is a constant function over an interval I:
For all a, b in I, if $a \leq b$ then $f(a) = f(b)$

The graph of a function is a horizontal line over an interval I.

Variations(prerequisites)

In general, a function can have only one variation or more than one variation.

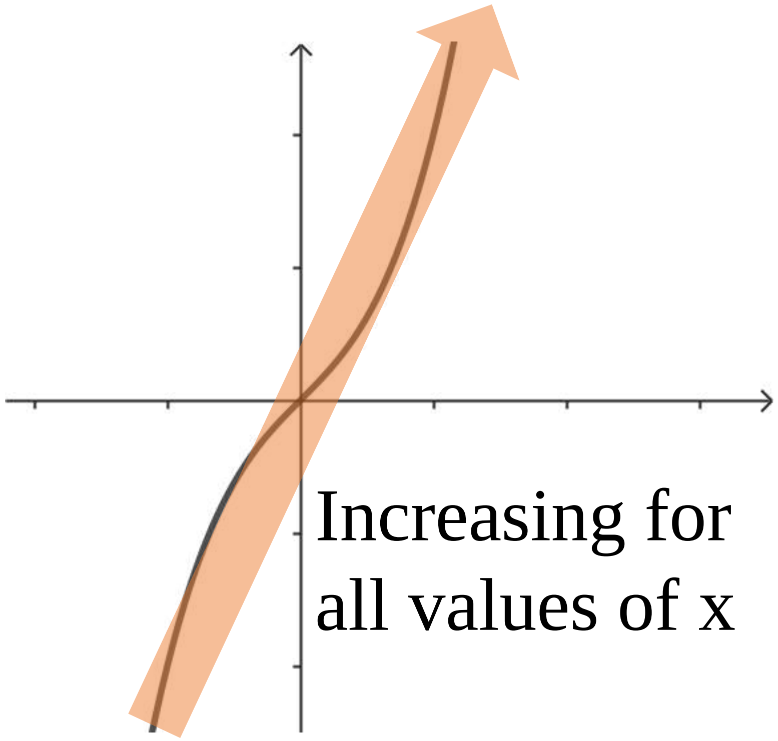
Example:



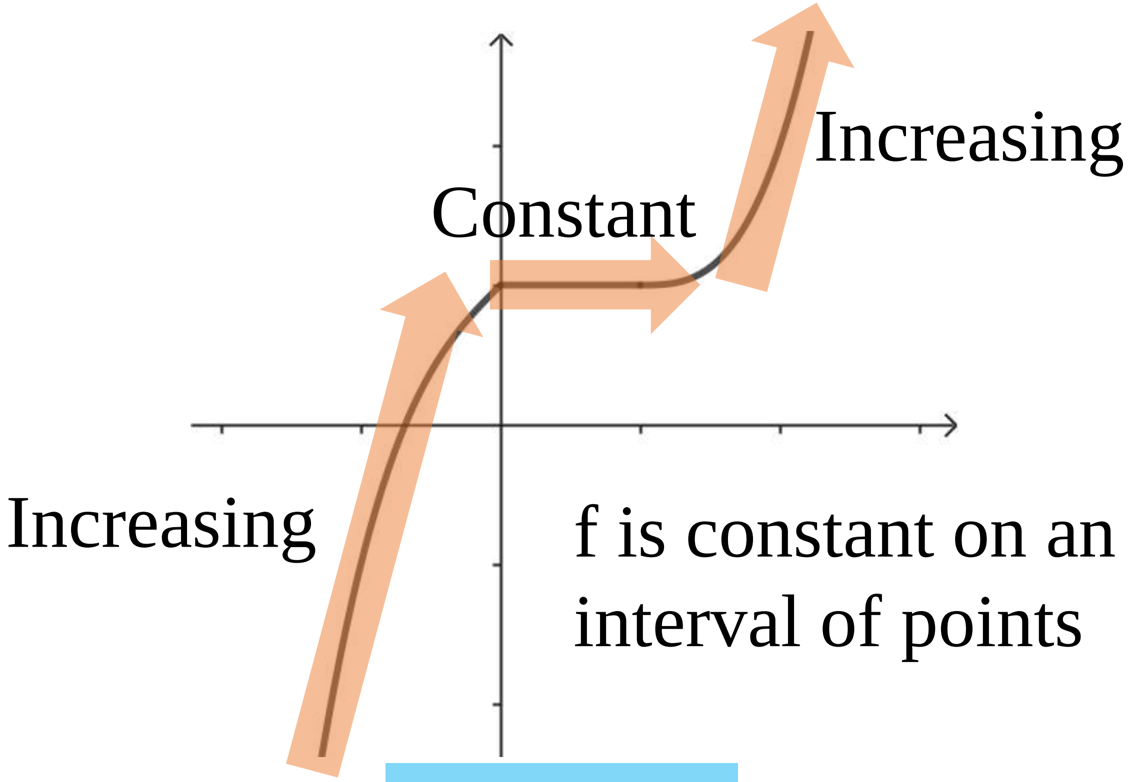
Variations(prerequisites)

What is the difference between increasing and strictly increasing?

Example:



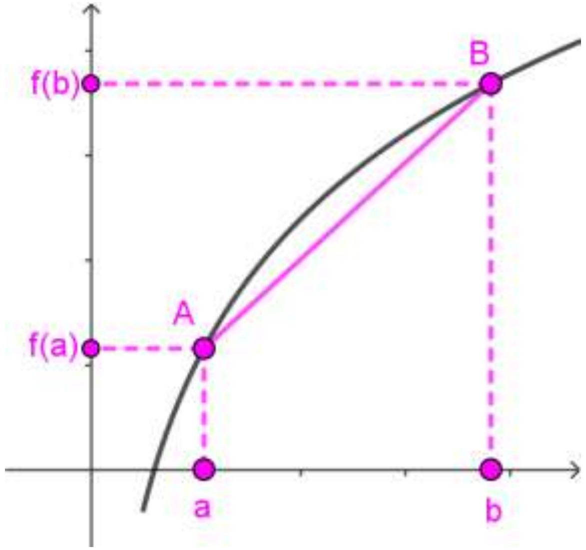
Strictly increasing

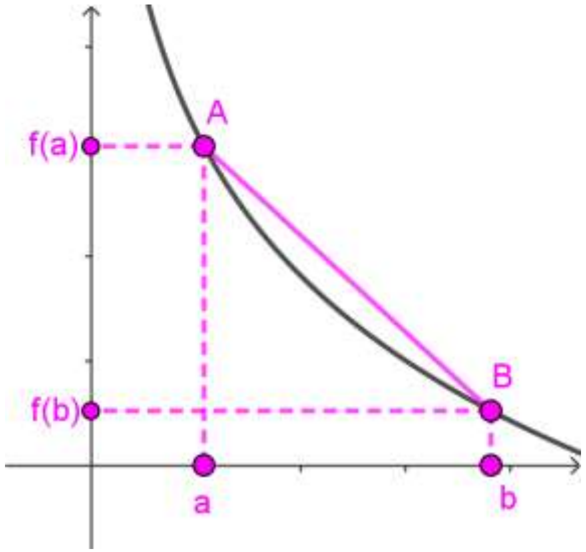


Increasing

Variations and derivative

What is the relation between the derived function f' of f and the variations of the function f?



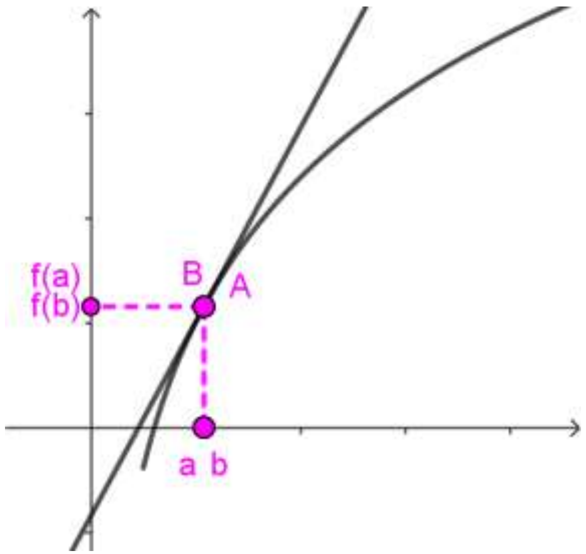


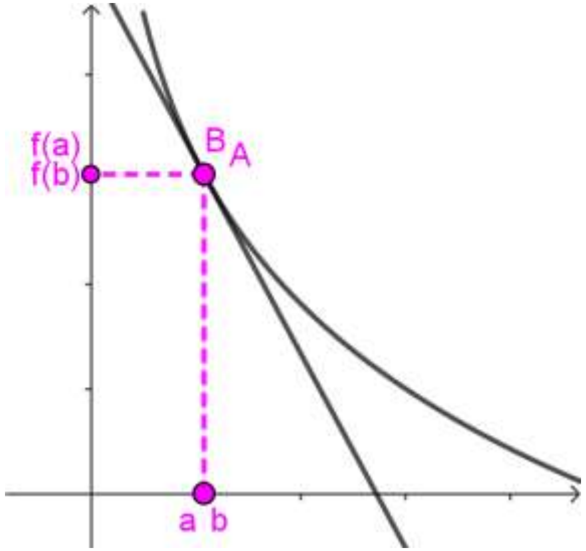
If the graph is increasing, the slope of the secant line (AB) is positive. $\frac{f(b)-f(a)}{b-a} > 0$

If the graph is decreasing, the slope of the secant line (AB) is negative. $\frac{f(b)-f(a)}{b-a} < 0$

Variations and derivative

Same for the slope of the tangent line when B becomes on A .





- ❖ f is increasing, the slope of the tangent is positive: derivative is positive.
- ❖ f decreasing, the slope of the tangent is negative: derivative is negative.

Variations and derivative

We can use derivative to study the variations of a function f:

1. Calculate $f'(x)$
2. Study the sign of $f'(x)$

3. Discuss:
   - If $f'(x) \geq 0$, the function is increasing.
   - If $f'(x) > 0$ except at some point, the function is strictly increasing
   - If $f'(x) \leq 0$, the function is decreasing.
   - If $f'(x) < 0$ except at some points, the function is strictly decreasing.

Variations and derivative

Example:

Consider the differentiable function $f(x) = x^2$ defined over IR.

$f'(x) = 2x$

- If $x > 0, f'(x) > 0$ so f is strictly increasing.
- If $x < 0, f'(x) < 0$ so f is strictly decreasing.
- If $x = 0\ , f'(x) = 0$ The tangent is horizontal

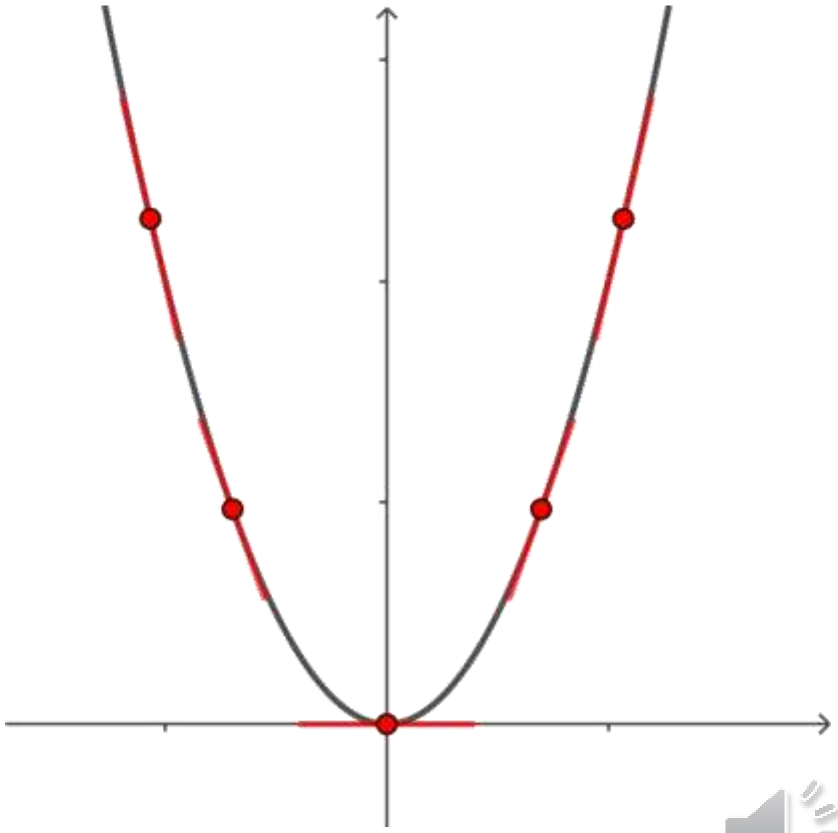


Table of variations

The table of variations summarizes the variations of a function f.

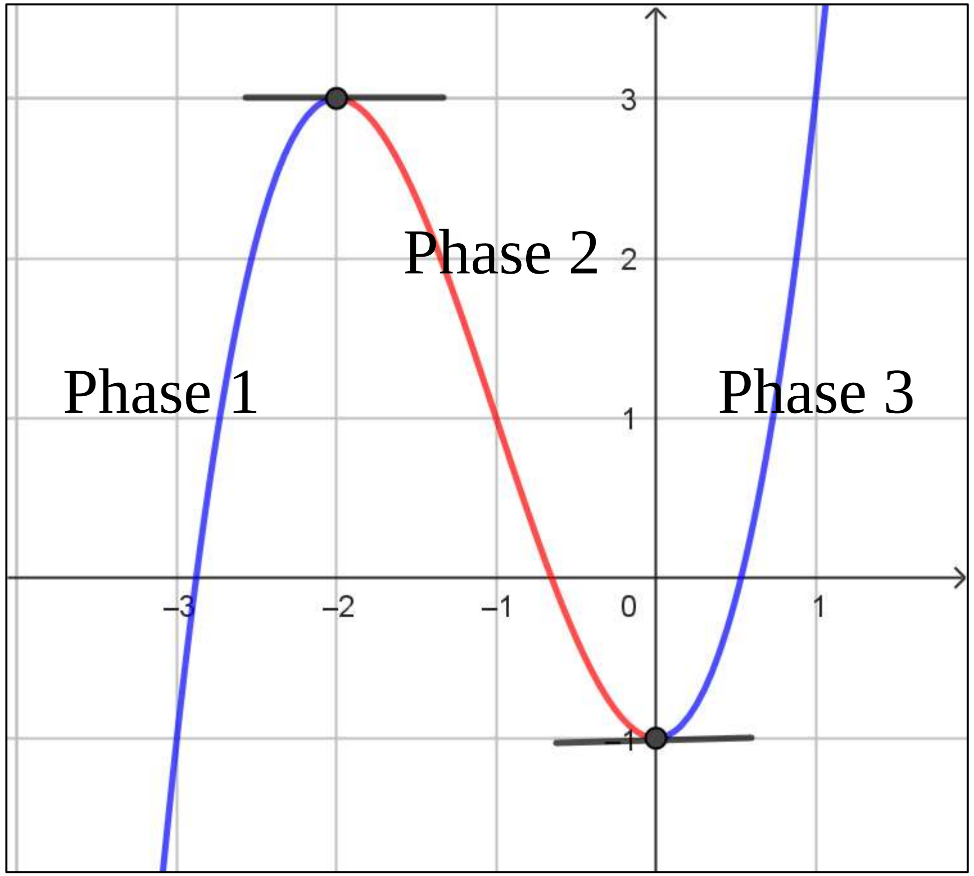
It includes:

- ❖ The domain of definition and the critical points.
- ❖ The sign of the derivative.
- ❖ The variations of the function.

x	Domain of definition and the critical points
$f'(x)$	Signs of the derivative
$f(x)$	Variations: ➚ : increasing ➘: decreasing

Table of variations

Example 1:



		Phase 1		Phase 2		Phase 3	
x	$-\infty$		-2		0		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	3	$\searrow$	-1	$\nearrow$	$+\infty$

Table of variations

Example 2:

Consider the differentiable function f defined over IR by $f(x) = x^2 + 2x - 3$

$f'(x) = 2x + 2$

$f'(x) = 0$; $2x + 2 = 0$; $x = -\frac{2}{2} = -1$

x	$-\infty$		-1		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	-4	$\nearrow$	$+\infty$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^2 = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2 = +\infty$$

$$f(-1) = (-1)^2 + 2(-1) - 3$$
$$= 1 - 2 - 3 = -4$$

Local extremum

Extremum: maximum or minimum

We said that a function f admits a local extremum at $x = a$ if:

- ❖ $f'(a) = 0$
- ❖ f' change its sign (f change its variations)

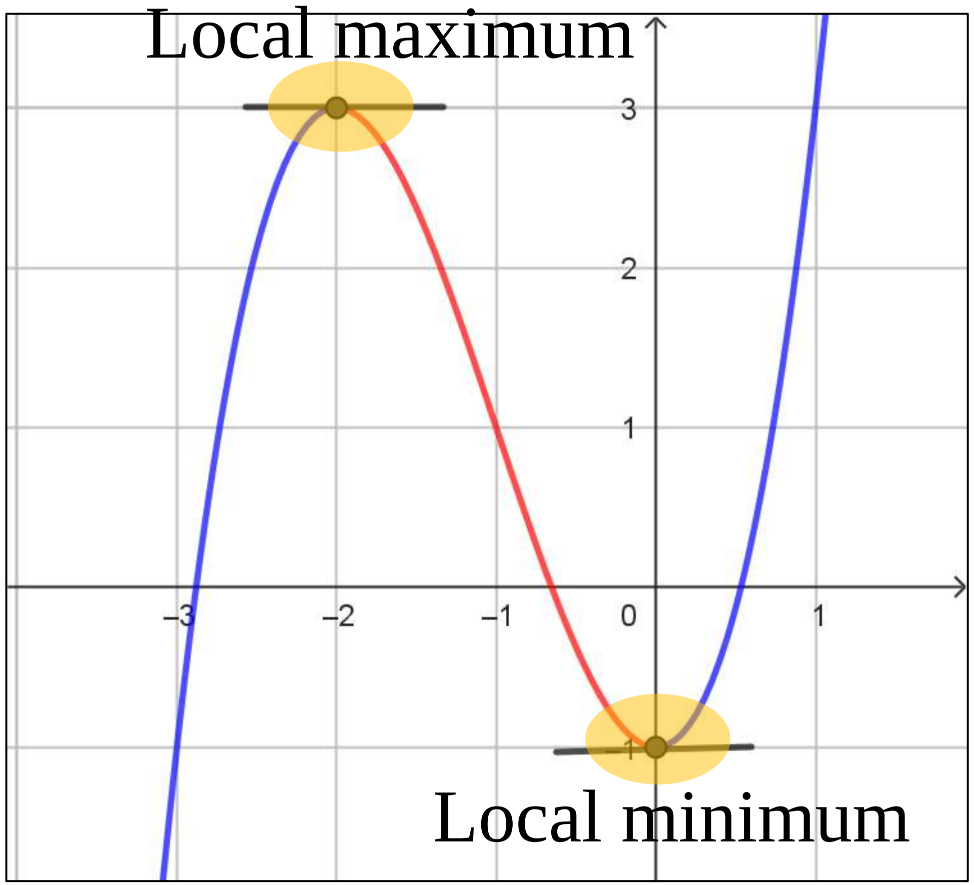
Example 1:

x	$-\infty$		-1		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	↘	-4	↗	$+\infty$

$f'(x)$ vanishes at $x = -1$ and changes its sign from – to +

Local extremum

Example 2:



Graphically:
Local extremum admits a horizontal tangent and the function changes its variation

Local extremum

Remark:

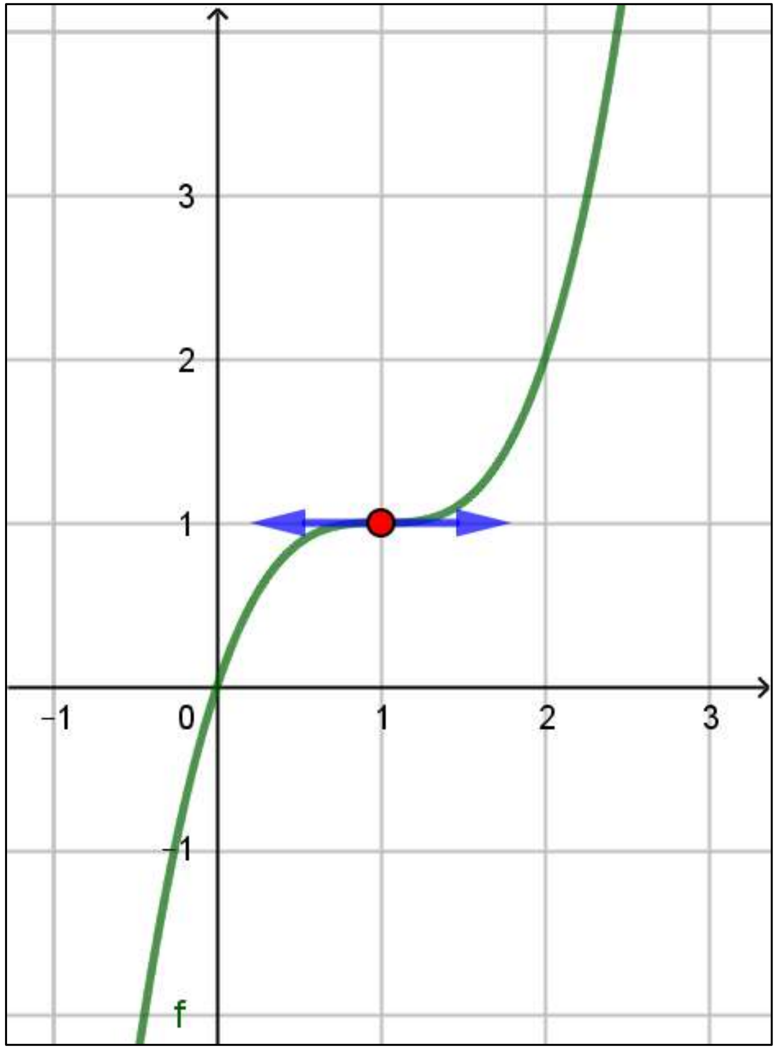
Given the function $f(x) = (x-1)^3 + 1$

$f'(x) = 3(x-1)^2 \geq 0$

$f'(x) = 0$ at $x = 1$

But $f'(x)$ doesn't change its sign at $x = 1$

So (1,1) doesn't represent a local maximum nor a local minimum.



Application 1

Consider the function f differentiable and defined over IR by $f(x) = x^3$.
Study the variations of f and set up its table of variations.

$f'(x) = 3x^2 \geq 0$ for all x

$f'(x) = 0$; $3x^2 = 0$; $x = 0$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^3 = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^3 = -\infty$$

$$f(0) = (0)^3 = 0$$

x	$-\infty$		0		$+\infty$
$f'(x)$		$+$	0	$+$	
$f(x)$	$-\infty$	↗	0	↗	$+\infty$

Application 2

The table of variations below represent a function f defined over IR.

x	$-\infty$		-1		0		2		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	0	$-$	
$f(x)$	$+\infty$	↘	-4	↗	5	↘	1	↘	2

Answer with true or false and justify.

1) The curve of f admits a horizontal asymptote.

True

$\lim_{x \to +\infty} f(x) = 2$ so $y = 2$ is a horizontal asymptote.

Application 2

The table of variations below represent a function f defined over IR.

x	$-\infty$		-1		0		2		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	0	$-$	
$f(x)$	$+\infty$	$\searrow$	-4	$\nearrow$	5	$\searrow$	1	$\searrow$	2

Answer with true or false and justify.

2) The curve of f admits 3 local extrema.

False

(-1;-4) is a local minimum , (0;5) is a local maximum but (2;1) is not a local extremum since f’ does not change its sign near to $x = 2$

Application 2

The table of variations below represent a function f defined over IR.

x	$-\infty$		-1		0		2		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	0	$-$	
$f(x)$	$+\infty$	$\searrow$	-4	$\nearrow$	5	$\searrow$	1	$\searrow$	2

Answer with true or false and justify.

3) The curve of f admits 3 horizontal tangents.

True

$f'(-1) = f'(0) = f'(2) = 0$ so the tangents at these three points are horizontal.

Application 2

The table of variations below represent a function f defined over IR.

x	$-\infty$		-1		0		2		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	0	$-$	
$f(x)$	$+\infty$	↘	-4	↗	5	↘	1	↘	2

Answer with true or false and justify.

4) $f'(-2) > 0$

False

When $x < -1$; $f'(x) < 0$;so $f'(-2) < 0$